

Lemaitre-Tolman-Bondi model of the universe and Hawking radiation of a dynamical horizon

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Abstract

The paper deals with anisotropic spherically symmetric Lemaitre-Tolman-Bondi model of the universe bounded by the apparent horizon. Using Hamilton-Jacobi method for both massive and massless test particles, we are able to show that the temperature associated with the apparent horizon due to Hawking radiation is the usual Hawking Temperature. Also the usual Lemaitre-Tolman-Bondi model and the transformed r-gauge Lemaitre-Tolman-Bondi model are shown to be equivalent. Finally, the results of Tunneling approach agree with the Hamiltonian-Jacobi method.

1 Introduction

Hawking radiation(HR), a quantum description of a black hole(BH), is intimately related to the existence of an event horizon of a BH. The discovery of HR [1] completes the cycle to describe BH as a thermodynamical object, i.e., BH behaves like a black body which emits thermal radiation with temperature proportional to its surface gravity and then Bekenstein [2] formulated the entropy proportional to the horizon area of the BH. This discovery also leads to view general relativity (GR) from different perspective. The interrelation between gravity and thermodynamics opens new avenues in GR. The motivation comes from the equivalence of Einstein equations with the first law of thermodynamics ($\delta Q = TdS$), first illustrated by Jacobson [3] and subsequently, it is extended to non-Einstein gravity [4, 5, 6, 7]. At present, it is commonly believed that gravity might be originated from the thermodynamics of the unknown microstructure of space-time. Although, a self-consistent full quantum theory of gravity is not yet formulated but still BH thermodynamics is playing the role of a bridge to combine general relativity with quantum mechanics.

The derivation of Hawking that "BH evaporates particles"[1, 8] was based on the quantum field theory. Subsequently, Hartle and Hawking [9] derived the BH temperature semiclassically, by using Feynmann path integral. Due to mathematical complexity of the above procedures, semiclassical approaches [10, 11, 12, 13] were developed for studying BH radiation. Basically, these semiclassical techniques were classified into two approaches – one, due to Parikh and Wilczek [10, 11] and the other developed by Padmanabhan et. al. [12, 13].

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In the first approach, the main ingredient is the consideration of energy conversion in tunneling of a thin shell from the hole and is known as the radial null geodesic method. Here the imaginary part of the action for the s-wave emission is related to the Boltzmann factor for emission to relate with Hawking temperature (HT). On the other hand, the second method is known as complex paths method. Here the action $S(r, t)$ for a single scalar particle is obtained by solving the Klein-Gordon (KG) equation in gravitational back ground using the Hamilton-Jacobi (HJ) approach. Then HT is obtained using the "The principle of detailed balance" [12, 13].

So far, most studies of HR effect deal with static space times, where there exist an event horizon. The global concept of event horizon is used to define the HT. But in dynamical space time, difficulty arises due to non existence of event horizon even locally. However, using the tunneling approach of Parikh-Wilczek [10], recently Hayward et. al. [14] proposed a locally defined HT for dynamical BH. Subsequently, Cai et. al. [15] have shown HR from the locally defined apparent horizon of the FRW universe where HT is measured by an observer using the Kodama vector [16, 17, 18] inside the horizon. Thus, at present one of the important issue in GR is to formulate the thermodynamics of the dynamical space times and its relation with gravity. Formulation of HR is one step forward towards this goal. In the present work, we deal with spherically symmetric LTB model of the space time and derive the HT using H-J method both for massless and massive particles. The semiclassical tunneling approach has been described and it results identical HT on the horizon. Finally, the paper ends with a brief discussion.

2 Lemaitre-Tolman-Bondi model and Hamilton-Jacobi method for massless particle

The inhomogeneous spherically symmetric LTB space-time model is described by the metric ansatz in a co-moving frame as

$$ds^2 = -dt^2 + \frac{R'^2}{1+f(r)}dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

where $R = R(r, t)$ is the area radius of the spherical surfaces and $f(r)(> -1)$ is the curvature scalar classifies the space-time as

- (i) bounded, if, $-1 < f(r) < 0$
- (ii) marginally bounded, if, $f(r) = 0$
- (iii) unbounded if $f(r) > 0$.

The Einstein field equations for the space time model can be written as [19, 20, 21, 22].

$$8\pi G\rho = \frac{F'(r, t)}{R^2 R'} \quad , \quad 8\pi Gp = -\frac{\dot{F}(r, t)}{R^2 \dot{R}} \quad (2)$$

and the evolution equation for R is

$$2R\ddot{R} + \dot{R}^2 + 8\pi GpR^2 = f(r). \quad (3)$$

Here the mass function

$$F(r, t) = R(\dot{R}^2 - f(r)), \quad (4)$$

is related to the mass contained within the co-moving radius r' .

The universe is assumed to fill with perfect fluid with energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \quad (5)$$

where the fluid 4-velocity u^μ has normalization $u^\mu u_\mu = -1$ and ρ, p are respectively the usual matter density and pressure of the fluid. The energy momentum conservation relation $T_{;\nu}^{\mu\nu} = 0$ demands the thermodynamic pressure to be homogeneous (i.e., $p' = 0$) and the evolution of matter density is given by

$$\dot{\rho} + 3H(\rho + p) = 0 \quad (6)$$

where, $H = \frac{1}{3} \left(\frac{\dot{R}'}{R'} + 2\frac{\dot{R}}{R} \right)$ is the Hubble parameter, '*dot*' and '*dash*' over any quantity stand for differentiation w.r.t. '*t*' and '*r*' respectively. Now splitting the metric ansatz (1) on the surface of the 2-sphere and on the 2-D hyper surface normal to the 2-sphere as

$$ds^2 = h_{ab}dx^a dx^b + R^2 d\Omega_2^2 \quad (7)$$

the dynamical apparent horizon is characterized by [23, 24]

$$h^{ab}\partial_a R \partial_b R = 0 \quad (8)$$

where $h_{ab} = \text{diag} \left(-1, \frac{R'^2}{1+f(r)} \right)$ is the 2-D metric normal to the 2-sphere. Thus the spherical surface of radius $R = R_A$ corresponding to the apparent horizon (i.e., marginally trapped surface) satisfies

$$R_A = F(r, t) \text{ and } \dot{R}_A^2 = 1 + f(r) \quad (9)$$

On the other hand, trapping horizon (R_τ), a hyper surface foliated by marginal spheres, is characterized by [23, 24]

$$\partial_+ R_\tau = 0, \quad \text{i.e.,} \quad \dot{R}_\tau = \sqrt{1 + f(r)} \quad (10)$$

where,

$$\partial_\pm = -\sqrt{2} \left(\partial_t \mp \frac{\sqrt{1+f(r)}}{R'} \partial_r \right), \quad (11)$$

are the null vectors normal to the 2-sphere.

Hence both the horizons coincide for LTB model as in FRW space-time.

Now, the Misner-Sharp gravitational mass (in units $G = 1$) is defined as [24, 25]

$$m(r, t) = \frac{R}{2} \left(1 - h^{ab} \partial_a R \partial_b R \right) \quad (12)$$

one may note that this mass ' m ' is an invariant quantity on the 2-D hyper surface normal to the 2-sphere and $m = m_H = \frac{R_A}{2}$ on the horizon. Further, one can introduce another invariant scalar associated with the horizon in the normal hyper surface, known as dynamic surface gravity and is defined as [14]

$$\kappa_D = \frac{1}{2\sqrt{-h}} \partial_a \left(\sqrt{-h} \ h^{ab} \partial_b R \right) |_H \quad (13)$$

which for the given model has the expression

$$\kappa_D = \left[\frac{1}{2R'} \left\{ -\partial_t (\dot{R}R') + \frac{1}{2} f'(r) \right\} \right]_H \quad (14)$$

For the LTB model with decomposed metric (7), the Kodama vector $\tilde{\kappa}$ has components

$$\kappa^a(r, t) = \frac{1}{\sqrt{-h}} \epsilon^{ab} \partial_b R \quad , \quad \kappa^\theta = 0 = \kappa^\phi \quad (15)$$

i.e.,

$$\kappa^\mu = \left(\sqrt{1+f(r)}, -\frac{\dot{R}}{R'} \sqrt{1+f(r)}, 0, 0 \right).$$

So $||\kappa^\mu||^2 = \dot{R}^2 - 1 - f(r)$, i.e., Kodama vector is time-like, null or space like for inside, on the surface or outside the apparent (i.e. trapping) horizon respectively. Note that Kodama vector is very similar to the time like Killing vector for stationary BH space time and consequently, an invariant energy associated with a particle is defined by the scalar on the normal space as [26]

$$\omega = -\kappa^a \partial_a I. \quad (16)$$

Here the classical action I of the massless particle satisfies the HJ equation

$$h^{ab} \partial_a I \partial_b I = 0 \quad (17)$$

$$i.e., \quad \left(\frac{\partial I}{\partial t} \right)^2 - \left\{ \frac{1+f(r)}{R'^2} \right\} \left(\frac{\partial I}{\partial r} \right)^2 = 0$$

Hence solving (16) and (17) we have

$$\frac{\partial I}{\partial r} = \frac{\omega R'}{\sqrt{1+f(r)} \left\{ \dot{R} - \sqrt{1+f(r)} \right\}} \quad and \quad \frac{\partial I}{\partial t} = \frac{\omega}{\left\{ \dot{R} - \sqrt{1+f(r)} \right\}} \quad (18)$$

where there is a pole at the horizon.

The full classical action of an outgoing massless particle is

$$I = \int_\gamma \partial_a I dx^a \quad (19)$$

where γ is an oriented curve with positive orientation along the increasing values of $x^a = (t, r)$. As for massless particles the radial motion is along a null direction so from the metric we have

$$0 = ds^2 = -dt^2 + \frac{R'^2}{1+f(r)} dr^2$$

i.e.

$$dt = \pm \frac{R'}{1+f(r)} dr, \quad (20)$$

for outgoing/ingoing particles respectively. Hence using equation (18) and (20) in the equation (19) we have for outgoing particles

$$I = 2 \int dr (\partial_r I) = -2 \int_\gamma \frac{\omega R'}{\{1+f(r)\} \left\{ 1 - \frac{\dot{R}}{\sqrt{1+f(r)}} \right\}} \quad (21)$$

Now expanding $G(r, t) = 1 - \frac{\dot{R}}{\sqrt{1+f(r)}}$ in the neighborhood of the horizon along a null direction we obtain [27]

$$G(r, t) \simeq \left[-\frac{\ddot{R}}{\sqrt{1+f}} \Delta t - \left(\frac{\dot{R}'}{\sqrt{1+f}} - \frac{\dot{R} f'(r)}{2(1+f)^{\frac{3}{2}}} \right) \Delta r \right]_H + \dots$$

$$\begin{aligned}
&= \left[-\frac{\ddot{R}R'}{(1+f)} - \frac{\dot{R}'}{\sqrt{1+f}} + \frac{\dot{R}f'(r)}{2(1+f)^{\frac{3}{2}}} \right] \Delta r|_H + \dots \\
&= \frac{2R'}{(1+f)} \kappa_D (r - r_H) + \dots
\end{aligned} \tag{22}$$

Then from (21)

$$I = - \int_{\gamma} dr \frac{\omega}{\kappa_D (r - r_H - i0)}, \tag{23}$$

which has a simple pole at $r = r_H$. Using Feynmann's $i\epsilon$ -prescription, the imaginary part of the action (the real part has no physical consequence) can be written as [27]

$$Im I = -\frac{\pi \omega_H}{\kappa_D} \tag{24}$$

Hence, one may interpret $T = -\frac{\kappa_D}{2\pi} > 0$ as the dynamical temperature associated with LTB space-times.

Moreover, we consider LTB tunneling computation in the co-ordinate system $(\tilde{r}, t, \theta, \phi)$ where $\tilde{r} = R$. The metric ansatz (in r -gauge) becomes

$$\begin{aligned}
ds^2 &= -Adt^2 - 2Bd\tilde{r}dt + C(d\tilde{r})^2 + \tilde{r}^2 d\Omega_2^2 \\
&= q_{ab} dx^a dx^b + \tilde{r}^2 d\Omega_2^2
\end{aligned} \tag{25}$$

where,

$$\begin{aligned}
A &= 1 - \frac{\dot{R}^2}{1+f(r)} = A(\tilde{r}, t) \\
B &= \frac{\dot{R}}{1+f(r)} = B(\tilde{r}, t) \\
\text{and } C &= \frac{1}{1+f(r)} = C(\tilde{r}, t)
\end{aligned}$$

the horizon is located at $\chi = 0$ where

$$\chi = q^{ab} \partial_a \tilde{r} \partial_b \tilde{r} = q^{\tilde{r}\tilde{r}} = \frac{A}{AC + B^2} \tag{26}$$

i.e., $A_H = 0$ gives the horizon.

The Kodama vector is given by

$$\kappa^\mu = \left(\frac{1}{\sqrt{B^2 + AC}}, \vec{0} \right) \tag{27}$$

and the invariant energy has the expression,

$$\omega = -\frac{\partial_t I}{\sqrt{B^2 + AC}} \tag{28}$$

The dynamical surface gravity on the horizon is

$$\kappa_D = \left[\frac{1}{2B^3} \left(A'B + \frac{1}{2} \dot{A}C \right) \right]_H \tag{29}$$

The HJ equation for a massless particle along a radial trajectory has the explicit form

$$-C(\partial_t I)^2 - 2B(\partial_t I)(\partial_{\tilde{r}} I) + A(\partial_{\tilde{r}} I)^2 = 0 \tag{30}$$

Now integration over the temporal coordinate gives real contribution to the particle action (which has no physical significance), an imaginary contribution comes only from integration along the radial direction. So, eliminating $\partial_t I$ between equation (28) and (30) we obtain

$$\partial_{\tilde{r}} I = -\frac{\omega (B + \sqrt{B^2 + AC})}{A} \sqrt{B^2 + AC} \quad (31)$$

For the metric (25), by making a null expansion on the horizon we obtain

$$0 = ds^2 = -2dt d\tilde{r} + \frac{d\tilde{r}^2}{\sqrt{1+f(r)}}$$

and

$$\begin{aligned} \left(1 - \frac{\dot{R}^2}{1+f(r)}\right) &\simeq -\frac{2\dot{R}\ddot{R}dt}{1+f(r)} - \left\{ \frac{2\dot{R}\dot{R}'}{1+f(r)} - \frac{f'(r)\dot{R}^2}{(1+f(r))^2} \right\} d\tilde{r} \\ &= -\left\{ \frac{\dot{R}\ddot{R}}{(1+f(r))^{\frac{3}{2}}} + \frac{2\dot{R}\dot{R}'}{1+f(r)} - \frac{f'(r)\dot{R}^2}{(1+f(r))^2} \right\} d\tilde{r} \\ &= 2\kappa_{DH} (\tilde{r} - \tilde{r}_H) \end{aligned} \quad (32)$$

Hence

$$Im I = -\int_{\gamma} d\tilde{r} \frac{\omega (B + \sqrt{B^2 + AC})}{2\kappa_{DH} (\tilde{r} - \tilde{r}_H - i0)} = -\frac{\pi\omega_H}{\kappa_{DH}} \quad (33)$$

Thus we have the same result.

3 Massive particle

We consider a particle of mass m' moving radially in the space time described by the metric (25). The HJ equation is

$$g^{\mu\nu} \partial_{\mu} S \partial_{\nu} S + m^2 = 0 \quad (34)$$

where the action S can be written as

$$S = \int \left(\frac{\partial S}{\partial t} \right) dt + \int \kappa_{\tilde{r}} d\tilde{r} \quad (35)$$

Here, $\kappa_{\tilde{r}}$, the radial momentum associated with the test particle and from equation (28), $\frac{\partial S}{\partial t}$ is related to the invariant energy ω as

$$\frac{\partial S}{\partial t} = -\omega \sqrt{B^2 + AC} \quad (36)$$

hence we write,

$$S = -\int \omega \sqrt{B^2 + AC} dt + \int \kappa_{\tilde{r}} d\tilde{r} \quad (37)$$

Now, the explicit form of the HJ equation (34) for the action (37) and the metric (25) is given by

$$\frac{-C}{(AC + B^2)} \omega^2 (AC + B^2) - \frac{2B}{AC + B^2} \omega \sqrt{AC + B^2} \kappa_{\tilde{r}} + \frac{A}{AC + B^2} \kappa_{\tilde{r}}^2 + m^2 = 0$$

i.e.,

$$A\kappa_{\tilde{r}}^2 - 2\omega\kappa_{\tilde{r}}B\sqrt{B^2 + AC} + (B^2 + AC)(m^2 - C\omega^2) = 0$$

which gives

$$\kappa_{\tilde{r}} = \frac{\sqrt{B^2 + AC}}{A} \left\{ \omega B \pm \sqrt{\omega^2 (B^2 + AC) - Am^2} \right\} \quad (38)$$

As the observer is inside the apparent horizon so considering the incoming mode (negative sign in the expression for κ_r ; positive sign corresponds to an outgoing mode), the imaginary part of the action is

$$ImS = Im \int \frac{\sqrt{B^2 + AC}}{A} \left\{ \omega B \pm \sqrt{\omega^2 (B^2 + AC) - Am^2} \right\} d\tilde{r} \quad (39)$$

It is to be noted that the integrand has a pole at the horizon, so performing the contour integration we have [15]

$$ImS = \pi\tilde{r}_H\omega \quad (40)$$

In tunneling approach, using WKB approximation the emission rate Γ (for particles tunnel from outside to inside the horizon) is the square of the tunneling amplitude [15]

$$i.e., \quad \Gamma \propto \exp\{-2ImS\} = \exp\{-2\pi\tilde{r}_H\omega\} \quad (41)$$

which can be casted in the form of a thermal spectrum $\Gamma \sim \exp\left\{-\frac{\omega}{T}\right\}$ provided $T = \frac{1}{2\pi\tilde{r}_H}$, the HT.

Thus an observer, inside the apparent horizon will see a thermal spectrum with HT when particles tunnel from outside to inside the horizon.

4 Tunneling approach

The basic idea in tunneling method is that following the standard approach, i.e., in the semiclassical approximation (i.e., WKB approximation), the emission rate of tunneling of a massless particle across the horizon can be related to the imaginary part of the action of the system. In the s-wave approximation, the particles are considered as massless shells, moving along a radial null geodesic.

So for the metric (25) the radial null geodesic is characterized by

$$\dot{\tilde{r}} = \frac{B \pm \sqrt{B^2 + AC}}{C} \quad (42)$$

where as before '+' or '-' sign indicates an outgoing or incoming null geodesic. Due to the tunneling of the particles from the outside to the inside the horizon, here we consider only an incoming geodesic. As we have seen that in the present context we need the imaginary part of the action produced by the tunneling particles (remaining part is always real) i.e., particles tunneling through a barrier (the classically forbidden region) so we obtain [15]

$$ImS = Im \int_{\tilde{r}_i}^{\tilde{r}_f} p_{\tilde{r}} d\tilde{r} = Im \int_{\tilde{r}_c}^{\tilde{r}_f} \int_0^{p_{\tilde{r}}} dp'_{\tilde{r}} d\tilde{r} \quad (43)$$

Here, the particle with radial momentum $p_{\tilde{r}}$ tunnels from the initial position \tilde{r}_i just outside the horizon to the final point at \tilde{r}_f which is a classical turning point, i.e., in the semiclassical analysis, the trajectory can represent a classically allowed motion

$$\dot{\tilde{r}} = \frac{d\tilde{H}}{dp_{\tilde{r}}} \Big|_{\tilde{r}} \quad (44)$$

where the Hamiltonian \tilde{H} is the generator of the cosmic time ' t' '. Then using equation (44) in equation (43) we write [15]

$$ImS = Im \int_{\tilde{r}_i}^{\tilde{r}_f} d\tilde{r} \int \frac{d\tilde{H}}{\dot{\tilde{r}}} = Im \int_{\tilde{r}_i}^{\tilde{r}_f} \frac{d\tilde{r}}{\dot{\tilde{r}}} \omega \sqrt{B^2 + AC} = -\omega Im \int_{\tilde{r}_i}^{\tilde{r}_f} \frac{d\tilde{r} C \sqrt{B^2 + AC}}{\{\sqrt{B^2 + AC} - B\}} = \pi \tilde{r}_H \omega \quad (45)$$

One may note that to perform the integration over the Hamiltonian H we get the energy as $\omega \sqrt{B^2 + AC}$, as measured by an observer with the Kodama vector. Then using the emission rate (41) we have

$$T = \frac{\omega}{2ImS} = \frac{1}{2\pi \tilde{r}_H}, \quad (46)$$

which is the HT as derived in the previous section.

5 Discussion

The paper deals with thermodynamics of the inhomogeneous LTB model of space-time. In the HJ formalism, we have considered the tunneling of both massless and massive particles, and the HT is measured by an observer with the Kodama vector (equation(15)) inside the apparent horizon (the trapping horizon). Thus apparent horizon of LTB universe is related to the HT in contrast to the BH case when HR is associated with the event horizon. Also the present work supports the results of FRW universe [15] in the literature. Also making a co-ordinate transformation (given by the metric equation (25)) we reach to the same conclusion, indicating HR independent of coordinate choice. Subsequently, tunneling approach (proposed by Parikh and Wilczek) has been described for massless particle to show the equivalence between the two semiclassical approaches. Finally, it will be interesting to improve our results by taking into account of back reaction of HR as in case of BH, consideration of back reaction results in a non-thermal spectrum of HR and as a result the issue of information loss may be overcome. Also for future work, it will be interesting to consider higher order correction terms and examine whether quantum correction to Hawking temperature [28] may be obtained.

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